# 1. Binomial theorem extended: Introduction

1. Expand the following in ascending powers of $x$ up to and including the term in $x^2$. State the range of $x$ for which the expansion is valid.
(a) $(1+x)^{-3}$
(b) $(1-x)^{-6}$
(c) $(1-2x)^{-3}$

2. Find the expansion of the following in ascending powers of $x$ up to and including the term in $x^2$. State the range of $x$ for which the expansion is valid.
(a) $(1+x)^{\frac{1}{3}}$
(b) $(1-x)^{-\frac{1}{2}}$
(c) $(1-\frac{1}{2}x)^{-\frac{1}{4}}$

3. $f(x)=\sqrt{1+3x}$, $-\frac{1}{3}<x<\frac{1}{3}$.
(a) Find the series expansion of $f(x)$, in ascending powers of $x$, up to and including the $x^3$ term. Simplify each term.
(b) Show that, when $x=\frac{1}{100}$, the exact value of $f(x)$ is $\frac{\sqrt{103}}{10}$.
(c) Find the percentage error made in using the series expansion in part (a) to estimate the value of $f(0.01)$. Give your answer to 2 significant figures.

4. (a) Find the binomial expansion of $(1-3x)^{\frac{3}{2}}$ in ascending powers of $x$ up to and including the $x^3$ term, simplifying each term.
(b) Show that, when $x=\frac{1}{100}$, the exact value of $(1-3x)^{\frac{3}{2}}$ is $\frac{97\sqrt{97}}{1000}$

(c) Substitute $x=\frac{1}{100}$ into the binomial expansion in part (a) and hence obtain an approximation to $\sqrt{97}$. Give your answer to 5 decimal places.

5. Show that $\dfrac{1}{(1-\frac{3}{2}x)^2}\approx 1+3x+\frac{27}{4}x^2+\frac{27}{2}x^3$ and state the interval of values of $x$ for which the expansion is valid. Use this to deduce the first four terms in the expansions of the following.
(a) $\dfrac{4}{(1-\frac{3}{2})^2}$
(b) $\dfrac{1}{(2-3x)^2}$

6*. $h(x)=\left(1+\frac{1}{x}\right)^{-\frac{1}{2}}$, $-1<x<1$
(a) Find the binomial expansion of $h(x)$ in ascending powers of $x$ up to and including the $x^2$ term, simplifying each term.
(b) Show that, when $x=9$, the exact value of $h(x)$ is $\frac{3\sqrt{10}}{10}$

(c) Use the expansion in part (a) to find an approximate value of $\sqrt{10}$. Write your answer to 2 decimal places.

7**. $\sqrt{2}$ can be approximated from $(1-2x)^{\frac{1}{2}}$ by setting $x=0.01$. Use this expansion, or other expansions to investigate finding approximations to $\sqrt{3}$, $\sqrt{5}$ and other prime numbers.

## 4. Binomial theorem extended: Introduction

1. Expand the following in ascending powers of $x$ up to and including the term in $x^2$. State the range of $x$ for which the expansion is valid.
(a) $(1+x)^{-3}$ $1-3x+6x^2$. Valid for $-1<x<1$
(b) $(1-x)^{-6}$ $1+6x+21x^2$. Valid for $-1<x<1$
(c) $(1-2x)^{-3}$ $1+6x+24x^2$. Valid for $-\frac{1}{2}<x<\frac{1}{2}$

2. Find the expansion of the following in ascending powers of $x$ up to and including the term in $x^2$. State the range of $x$ for which the expansion is valid.
(a) $(1+x)^{\frac{1}{3}}$ $1+\frac{1}{3}x-\frac{1}{9}x^2$. Valid for $-1<x<1$
(b) $(1-x)^{-\frac{1}{2}}$ $1+\frac{1}{2}x+\frac{3}{8}x^2$. Valid for $-1<x<1$
(c) $(1-\frac{1}{2}x)^{-\frac{1}{4}}$ $1+\frac{1}{8}x+\frac{5}{128}x^2$. Valid for $-2<x<2$

3. $f(x)=\sqrt{1+3x}$, $-\frac{1}{3}<x<\frac{1}{3}$.
(a) Find the series expansion of $f(x)$, in ascending powers of $x$, up to and including the $x^3$ term. Simplify each term. $1+\frac{3}{2}x-\frac{9}{8}x^2+\frac{27}{16}x^3$
(b) Show that, when $x=\frac{1}{100}$, the exact value of $f(x)$ is $\frac{\sqrt{103}}{10}$. $\sqrt{1.03}=\sqrt{\frac{103}{100}}=\frac{\sqrt{103}}{10}$
(c) Find the percentage error made in using the series expansion in part (a) to estimate the value of $f(0.01)$. Give your answer to 2 significant figures. $3.1\times10^{-6}\%$

4. (a) Find the binomial expansion of $(1-3x)^{\frac{3}{2}}$ in ascending powers of $x$ up to and including the $x^3$ term, simplifying each term. $1-\frac{9}{2}x+\frac{27}{8}x^2+\frac{27}{16}x^3$
(b) Show that, when $x=\frac{1}{100}$, the exact value of $(1-3x)^{\frac{3}{2}}$ is $\frac{97\sqrt{97}}{1000}$ $0.97^{\frac{3}{2}}=0.97\times0.97^{\frac{1}{2}}=\frac{97}{100}\times\sqrt{\frac{97}{100}}=\frac{97}{100}\times\frac{\sqrt{97}}{10}=\frac{97\sqrt{97}}{1000}$
(c) Substitute $x=\frac{1}{100}$ into the binomial expansion in part (a) and hence obtain an approximation to $\sqrt{97}$. Give your answer to 5 decimal places. 9.84886

5. Show that $\dfrac{1}{(1-\frac{3}{2}x)^2}\approx1+3x+\frac{27}{4}x^2+\frac{27}{2}x^3$ and state the interval of values of $x$ for which the expansion is valid. $|x|<\frac{2}{3}$ Use this to deduce the first four terms in the expansions of the following.
(a) $\dfrac{4}{(1-\frac{3}{2})^2}$ $4+12x+27x^2+54x^3$
(b) $\dfrac{1}{(2-3x)^2}$ $\frac{1}{4}+\frac{3}{4}x+\frac{27}{16}x^2+\frac{27}{8}x^3$

6*. $h(x)=\left(1+\frac{1}{x}\right)^{-\frac{1}{2}}$, $-1<x<1$
(a) Find the binomial expansion of $h(x)$ in ascending powers of $x$ up to and including the $x^2$ term, simplifying each term. $1-\frac{1}{2x}+\frac{3}{8x^2}$
(b) Show that, when $x=9$, the exact value of $h(x)$ is $\frac{3\sqrt{10}}{10}$ $h(x)=\left(1+\frac{1}{9}\right)^{-\frac{1}{2}}=\left(\frac{10}{9}\right)^{-\frac{1}{2}}=\sqrt{\frac{9}{10}}=\frac{3}{\sqrt{10}}=\frac{3\sqrt{10}}{10}$
(c) Use the expansion in part (a) to find an approximate value of $\sqrt{10}$. Write your answer to 2 decimal places. 3.16

7**. $\sqrt{2}$ can be approximated from $(1-2x)^{\frac{1}{2}}$ by setting $x=0.01$. Use this expansion, or other expansions to investigate finding approximations to $\sqrt{3}$, $\sqrt{5}$ and other prime numbers.