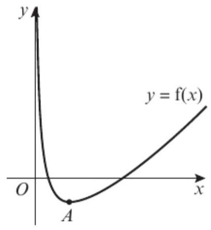
4. Solving equations numerically: The Newton–Raphson method

1. $f(x) = x^3 - 2x - 1$
(a) Show that the equation $f(x) = 0$ has a root, α, in the interval $1 < \alpha < 2$.

(b) Using $x_0 = 1.5$ as a first approximation to α, apply the Newton–Raphson procedure once to $f(x)$ to find a second approximation to α, giving your answer to 3 decimal places.

2. The diagram shows part of the curve with equation
$y = f(x)$, where $f(x) = x^{\frac{3}{2}} - e^{-x} + \frac{1}{\sqrt{x}} - 2, x > 0$
The point A, with x-coordinate q, is a stationary point on the curve.
The equation $f(x) = 0$ has a root α in the interval $[1.2, 1.3]$.
(a) Explain why $x_0 = q$ is not suitable to use as a first approximation when applying the Newton–Raphson method.



(b) Taking $x_0 = 1.2$ as a first approximation to α, apply the Newton–Raphson process once to $f(x)$ to obtain a second approximation to α. Give your answer to 3 decimal places.

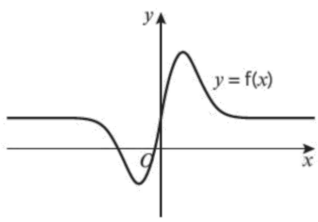
3. $f(x) = 1 - x - \cos(x^2)$
(a) Show that the equation $f(x) = 0$ has a root α in the interval $1.4 < \alpha < 1.5$.

(b) Using $x_0 = 1.4$ as a first approximation to α, apply the Newton–Raphson method once to $f(x)$ to find a second approximation to α, giving your answer to 3 decimal places.

(c) By considering the change of sign of $f(x)$ over an appropriate interval, show that your answer to part (b) is correct to 3 decimal places.

4. $y = f(x)$, where $f(x) = x^2 \sin(x) - 2x + 1$. A root lies between 2.4 and 2.5. Using $x_0 = 2.4$ as a first approximation, apply the Newton–Raphson method to $f(x)$ to obtain a second approximation. Give your answer to 3 decimal places.

*5. $f(x) = \frac{1}{5} + xe^{-x^2}$
The diagram shows a sketch of the curve $y = f(x)$. The curve has a horizontal asymptote at $y = \frac{1}{5}$.
(a) Prove that the Newton–Raphson method will fail to converge on a root of $f(x) = 0$ for all values of $x_0 > \frac{1}{\sqrt{2}}$

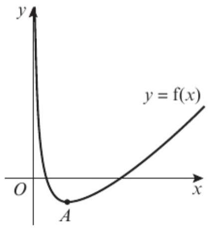


(b) Taking -0.5 as a first approximation, use the Newton–Raphson method to find the root of $f(x) = 0$ that lies in the interval $[-1, 0]$, giving your answer to 3 decimal places.

8. Solving equations numerically: The Newton–Raphson method

1. $f(x) = x^3 - 2x - 1$
(a) Show that the equation $f(x) = 0$ has a root, α, in the interval $1 < \alpha < 2$. $f(1) = -2$, $f(2) = 3$. There is a change of sign (and state that $f(x)$ is continuous over the interval)
(b) Using $x_0 = 1.5$ as a first approximation to α, apply the Newton–Raphson procedure once to $f(x)$ to find a second approximation to α, giving your answer to 3 decimal places. $x_1 = 1.632$

2. The diagram shows part of the curve with equation
$y = f(x)$, where $f(x) = x^{\frac{3}{2}} - e^{-x} + \dfrac{1}{\sqrt{x}} - 2, x > 0$

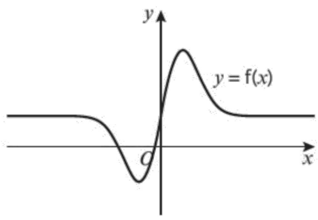


The point A, with x-coordinate q, is a stationary point on the curve.
The equation $f(x) = 0$ has a root α in the interval $[1.2, 1.3]$.
(a) Explain why $x_0 = q$ is not suitable to use as a first approximation when applying the Newton–Raphson method. A is a stationary point on the curve so $f'(q) = 0$. It is not possible to divide by zero using the Newton–Raphson method, so this value of x_0 cannot be used.
(b) Taking $x_0 = 1.2$ as a first approximation to α, apply the Newton–Raphson process once to $f(x)$ to obtain a second approximation to α. Give your answer to 3 decimal places. $f'(x) = \frac{3}{2}x^{\frac{1}{2}} + e^{-x} - \dfrac{1}{2x^{\frac{3}{2}}}$ so $x_1 = 1.247$

3. $f(x) = 1 - x - \cos(x^2)$
(a) Show that the equation $f(x) = 0$ has a root α in the interval $1.4 < \alpha < 1.5$. $f(1.4) = 0.0205...$, $f(1.5 = 0.128....$ There is a change of sign (and state that $f(x)$ is continuous over the interval)
(b) Using $x_0 = 1.4$ as a first approximation to α, apply the Newton–Raphson method once to $f(x)$ to find a second approximation to α, giving your answer to 3 decimal places. $f'(x) = -1 + 2x\sin(x^2)$, $x_1 = 1.413$
(c) By considering the change of sign of $f(x)$ over an appropriate interval, show that your answer to part (b) is correct to 3 decimal places. $f(1.4125) = 0.00076... < 0$, $f(1.4135) = 0.00081... > 0$. There is a sign change in the interval $[1.4125, 1.4135]$ so $x = 1.413$ is correct to 3 decimal places.

4. $y = f(x)$, where $f(x) = x^2\sin(x) - 2x + 1$. A root lies between 2.4 and 2.5. Using $x_0 = 2.4$ as a first approximation, apply the Newton–Raphson method to $f(x)$ to obtain a second approximation. Give your answer to 3 decimal places. $f'(x) = x^2\cos x + 2x\sin x - 2$, $x = 2.430$

*5. $f(x) = \frac{1}{5} + xe^{-x^2}$
The diagram shows a sketch of the curve $y = f(x)$. The curve has a horizontal asymptote at $y = \frac{1}{5}$.



(a) Prove that the Newton–Raphson method will fail to converge on a root of $f(x) = 0$ for all values of $x_0 > \frac{1}{\sqrt{2}}$ $f'(x) = e^{-x^2}(1-2x^2) = 0$ for stationary points so $1 - 2x^2 = 0$ so $x = \pm\frac{1}{\sqrt{2}}$. From the graph (or looking at $e^{-x^2}(1-2x^2)$), $f'(x) < 0$ for $x > \frac{1}{\sqrt{2}}$. $f(x) > 0$ for $x > \frac{1}{\sqrt{2}}$. Hence $x_{n+1} = x_n - \frac{f(x)}{f'(x)}$ will be an increasing sequence for $x > \frac{1}{\sqrt{2}}$ and will fail to converge.
(b) Taking -0.5 as a first approximation, use the Newton–Raphson method to find the root of $f(x) = 0$ that lies in the interval $[-1, 0]$, giving your answer to 3 decimal places. $x_1 = -0.0136...$, $x_2 = -0.2001...$, $x_3 = -0.2088...$, $x_4 = 0.2089....$ The root is -0.209 to 3 decimal places.