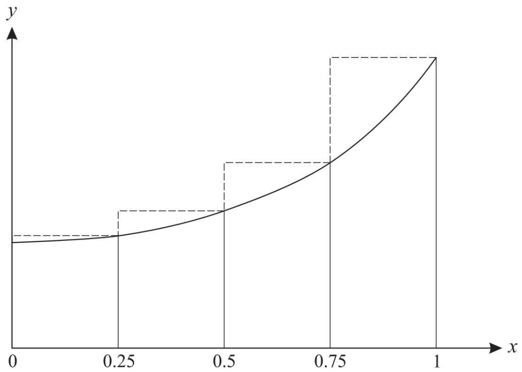
4. Approximating areas: The rectangle approximation

1. The diagram shows the curve with equation $y = e^{x^2}$, for $0 \le x \le 1$. The region under the curve between these limits is divided into four strips of equal width. The area of this region under the curve is A.

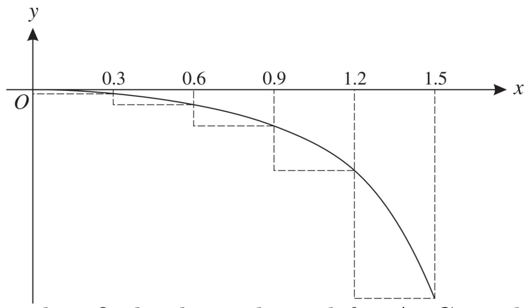
i) By considering the set of rectangles indicated in the diagram, show that an upper bound for A is 1.71.



(ii) By considering an appropriate set of four rectangles, find a lower bound for A.

2. The diagram shows the curve with equation $y = \ln(\cos x)$, for $0 \le x \le 1.5$. The region bounded by the curve, the x-axis and the line $x = 1.5$ has area A. The region is divided into five strips, each of width 0.3.

(i) By considering the set of rectangles indicated in the diagram, find an upper bound for A. Give the answer correct to 3 decimal places.

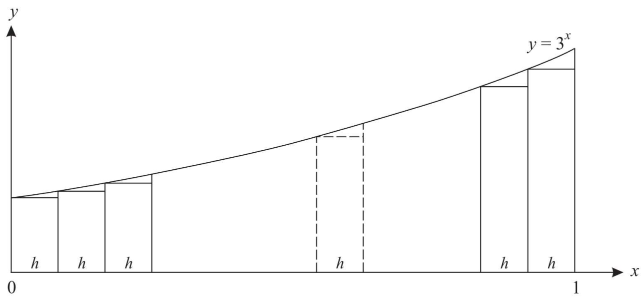


(ii) By considering another set of five suitable rectangles, find a lower bound for A. Give the answer correct to 3 decimal places.

(iii) How could you reduce the difference between the upper and lower bounds for A?

3. The diagram shows the curve with equation $y = 3^x$ for $0 \le x \le 1$. The area A under the curve between these limits is divided into n strips, each of width h where $nh = 1$.

(i) By using the set of rectangles indicated on the diagram, show that $A > \frac{2h}{3^h - 1}$



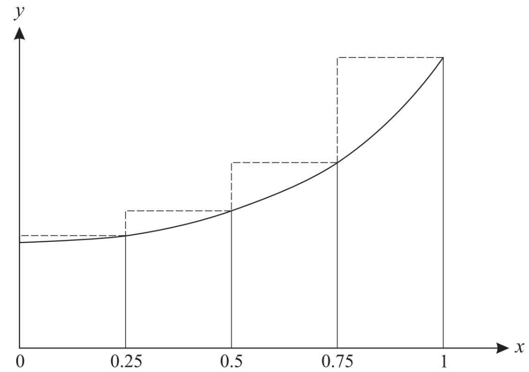
(ii) By considering another set of rectangles, show that $A < \frac{(2h)3^h}{3^h - 1}$.

(iii) Given that $h = 0.001$, use these inequalities to find values between which A lies.

(iv) By integrating exactly, generate an underestimate and overestimate for $\ln 3$.

8. Approximating areas: The rectangle approximation

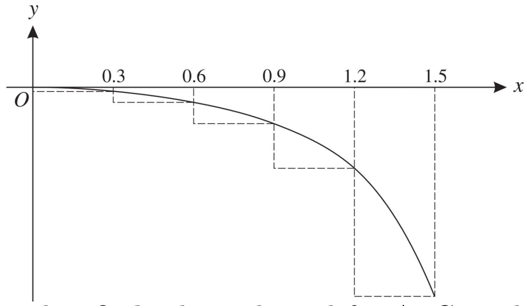
1. The diagram shows the curve with equation $y = e^{x^2}$, for $0 \le x \le 1$. The region under the curve between these limits is divided into four strips of equal width. The area of this region under the curve is A.



i) By considering the set of rectangles indicated in the diagram, show that an upper bound for A is 1.71. Show area is $\frac{1}{4}(e^{0.25^2} + e^{0.5^2} + e^{0.75^2} + e) = 1.7054$. The actual integral will be less than this so $A < 1.71$

(ii) By considering an appropriate set of four rectangles, find a lower bound for A. Left sum $= \frac{1}{4}(e^0 + e^{0.25^2} + e^{0.5^2} + e^{0.75^2}) = 1.2758$ so $A > 1.27$

2. The diagram shows the curve with equation $y = \ln(\cos x)$, for $0 \le x \le 1.5$. The region bounded by the curve, the x-axis and the line $x = 1.5$ has area A. The region is divided into five strips, each of width 0.3.

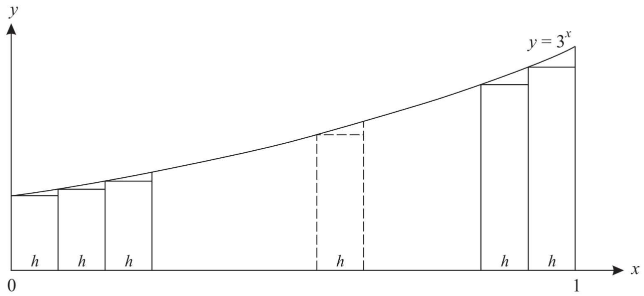


(i) By considering the set of rectangles indicated in the diagram, find an upper bound for A. Give the answer correct to 3 decimal places. 1.314

(ii) By considering another set of five suitable rectangles, find a lower bound for A. Give the answer correct to 3 decimal places. 0.518

(iii) How could you reduce the difference between the upper and lower bounds for A? Use more rectangles

3. The diagram shows the curve with equation $y = 3^x$ for $0 \le x \le 1$. The area A under the curve between these limits is divided into n strips, each of width h where $nh = 1$.



(i) By using the set of rectangles indicated on the diagram, show that $A > \frac{2h}{3^h - 1}$

Left sum $= h(3^0 + 3^h + 3^{2h} + ... + 3^{(n-1)h})$. This is a geometric sequence with n terms with $a = 1$ and $r = 3^h$, so left sum $= h \times \frac{3^{nh}-1}{3^h-1}$ which simplifies to $\frac{2h}{3^h-1}$ since $nh = 1$. This is an underestimate so $A > \frac{2h}{3^h-1}$

(ii) By considering another set of rectangles, show that $A < \frac{(2h)3^h}{3^h-1}$. Right sum $= h(3^h + 3^{2h} + ... + 3^{nh}) = 3^h \times h(3^0 + 3^h + 3^{2h} + ... + 3^{(n-1)h}) = 3^h \times \frac{2h}{3^h-1} = \frac{(2h)3^h}{3^h-1}$. This an overestimate so $A < \frac{(2h)3^h}{3^h-1}$

(iii) Given that $h = 0.001$, use these inequalities to find values between which A lies. Substituting $h = 0.001$ into the answers to (i) amd (ii) in gives $1.8194 \le A \le 1.8214$

(iv) By integrating exactly, generate an underestimate and overestimate for $\ln 3$. $\int_0^1 3^x \mathrm{d}x = \left[\frac{3^x}{\ln 3}\right]_0^1 = \frac{2}{\ln 3}$. Hence $1.8194 \le \frac{2}{\ln 3} \le 1.8214$ so $1.0980 < \ln 3 < 1.0992$. Note that the exact value is 1.0986. We could improve our estimate by making h smaller :-)