4. Proof: Proof by counter-example

1. Find a counter-example to disprove each of the following statements:
(a) If n is a positive integer then $n^4 - n$ is divisible by 4.
(b) Integers always have an even number of factors.
(c) $2n^2 - 6n + 1$ is positive for all values of n.
(d) $2n^2 - 2n - 4$ is a multiple of 3 for all integer values of n.

2. (a) Prove that for any positive numbers p and q that $p + q \geq \sqrt{4pq}$.
(b) Show, by means of a counter-example, that this inequality does not hold when p and q are both negative.

3. Prove that for all real values of x that $(x + 6)^2 \geq 2x + 11$.

4. It is claimed that the following inequality is true for all negative numbers x and y:
$x + y \geq \sqrt{x^2 + y^2}$
The following proof is offered by a student:

$x + y \geq \sqrt{x^2 + y^2}$
$(x + y)^2 \geq x^2 + y^2$
$x^2 + y^2 + 2xy \geq x^2 + y^2$
$2xy > 0$ which is true because x and y are both negative, so xy is positive.

(a) Explain the error made the student.
(b) By use of a counter-example, verify that the inequality is not satisfied if both x and y are negative.
(c) Prove that this inequality is true if x and y are both positive.

5. Read up on Fermat's last theorem; this website gives a good introduction: http://mathshistory.st-andrews.ac.uk/HistTopics/Fermat%27s_last_theorem.html. You may be interested in it from a purely mathematical perspective, or perhaps from seeing just how many people worked on trying to prove it. One thing that I find interesting is that this seemingly pure piece of number theory ended up having a deep connection with understanding space.

9. Proof: Proof by counter-example

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(a) Explain the error made the student.
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